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LATERAL DEFORMATION OF SHALLOW SHELLS OF REVOLUTION

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Abstract—Shallow shells of revolution of radially varying thickness undergoing small deformations proportional to the sine or cosine of the polar angle (lateral deformation) are considered. By use of certain integrals expressing overall equilibrium and compatibility of the shell, it is shown that the solution of the governing differential equations can be reduced to the solution of two coupled second-order differential equations. Explicit solutions for a class of shells whose meridional slope varies as a power of the radius are obtained in terms of Kelvin functions and their integrals.

INTRODUCTION

IN THIS paper, we consider shallow shells of revolution of radially varying elastic properties undergoing small deformations proportional to the sine or cosine of the polar angle (which here will be denoted as lateral deformations). It is shown that the solution to the two coupled fourth-order partial differential equations for the normal displacement W and Airy's stress function F which govern the problem for arbitrary deformations can be reduced to the solution of two coupled *second-order* ordinary differential equations. For shells of constant thickness and constant elastic properties whose meridional slope varies as a power of the radius, it is further shown that the solutions of the second-order equations can be expressed in terms of Kelvin functions and their integrals. An exception is the shell of logarithmic profile for which the solutions can be expressed in terms of elementary functions.

That the equations for arbitrary shells of revolution undergoing lateral deformations should admit of reductions similar to those known to exist for axi-symmetric deformations, was pointed out by Novozhilov [1]. For both axi-symmetric and lateral deformations, the reductions are possible because first, certain integrals expressing overall equilibrium and compatibility conditions of the shell become first integrals of the differential equations; and second, because rigid body displacements are solutions of the homogeneous differential equations. Utilizing these ideas, Chernin [2] has obtained a complex-valued second-order ordinary differential equation for the lateral deformation of arbitrary shells of revolution of constant thickness. While a number of our equations could be obtained from analogous ones of Chernin by an appropriate limiting process, we feel that it is simpler to develop these equations from the start within the framework of shallow shell theory. We also wish to indicate the special treatment that must be given to the Airy stress function F, which is a convenient and standard variable to use in shallow shell theory. Finally, we remark that our results are more extensive than Chernin's in that we allow for radially varying elastic properties and that we give closed form solutions for a class of shell profiles.

THE GOVERNING EQUATIONS

The equilibrium equations of the linear theory of shallow shells of revolution, expressed in terms of the direct stress resultants, $N_r, N_\theta, N_{r\theta}\,(=N_{\theta r})$, the stress couples $M_r, M_\theta, M_{r\theta}\,(=M_{\theta r})$, and the transverse shear stress resultants Q_r, Q_θ, are

$$(rN_r)' + N_{r\theta}^{\cdot} - N_\theta + rP_r = 0, \tag{1}$$

$$(rN_{r\theta})' + N_\theta^{\cdot} + N_{r\theta} + rP_\theta = 0, \tag{2}$$

$$(rQ_r)' + Q_\theta^{\cdot} + (rz'N_r)' + z'N_{r\theta}^{\cdot} + rP_z = 0, \tag{3}$$

$$(rM_r)' + M_{r\theta}^{\cdot} - M_\theta - rQ_r = 0, \tag{4}$$

$$(rM_{r\theta})' + M_\theta^{\cdot} + M_{r\theta} - rQ_\theta = 0, \tag{5}$$

where $z = z(r)$ is the midsurface profile equation, P_r, P_θ, and P_z are the components of the load intensity in the radial, circumferential, and axial directions, and primes and dots indicate differentiation with respect to r and θ, respectively.

The extensional and bending strains are expressed in terms of the radial, circumferential, and axial displacement components U, V, and W as follows

$$\varepsilon_r = U' + z'W, \qquad \varepsilon_\theta = \frac{U}{r} + \frac{V^{\cdot}}{r}, \tag{6a,b}$$

$$2\varepsilon_{r\theta} = r\left(\frac{V}{r}\right)' + \frac{U^{\cdot}}{r} + z'\frac{W^{\cdot}}{r}, \tag{6c}$$

$$\kappa_r = -W'', \qquad \kappa_\theta = -\frac{W'}{r} - \frac{W^{\cdot\cdot}}{r^2}, \qquad \kappa_{r\theta} = -\left(\frac{W^{\cdot}}{r}\right)'. \tag{7a,b,c}$$

Assuming isotropy, the stress–strain relations are

$$\varepsilon_r = A(N_r - \nu N_\theta), \qquad \varepsilon_\theta = A(N_\theta - \nu N_r), \qquad \varepsilon_{r\theta} = (1+\nu)AN_{r\theta}, \tag{8}$$

$$M_r = D(\kappa_r + \nu\kappa_\theta), \qquad M_\theta = D(\kappa_\theta + \nu\kappa_r), \qquad M_{r\theta} = (1-\nu)D\kappa_{r\theta}, \tag{9}$$

where

$$A = \frac{1}{Eh}, \qquad D = \frac{Eh^3}{12(1-\nu^2)},$$

and where E is Young's modulus, ν is Poisson's ratio, and h is the shell thickness.

It will be assumed that the external load terms P_r and P_θ are derivable from a load potential Φ,

$$P_r = -\Phi', \qquad P_\theta = r^{-1}\Phi^{\cdot}$$

in which case equations (1) and (2) can be satisfied identically in terms of a stress function F as follows:

$$N_r = \frac{F'}{r} + \frac{F^{\cdot\cdot}}{r^2} + \Phi, \qquad N_\theta = F'' + \Phi, \qquad N_{r\theta} = -\left(\frac{F^{\cdot}}{r}\right)'. \tag{10a,b,c}$$

Upon elimination of the transverse stress resultants, the remaining three equilibrium equations (3)–(5) reduce to

$$(rM_r)''+2r^{-1}M^{\cdot}_{r\theta}+2M'^{\cdot}_{r\theta}-M'_\theta+r^{-1}M^{\cdot\cdot}_\theta+(z'F)'+r^{-1}z''F^{\cdot\cdot} = -rP_z-(rz'\Phi)'. \quad (11)$$

As in the theory of plane stress, the introduction of the stress function F as a dependent variable requires the satisfaction of an extensional strain compatibility equation. For a shallow shell of revolution, this equation is, consistent with equations (6),

$$(r\varepsilon_\theta)''-2r^{-1}\varepsilon^{\cdot}_{r\theta}-2\varepsilon'^{\cdot}_{r\theta}-\varepsilon'_r+r^{-1}\varepsilon^{\cdot\cdot}_r+(z'W')'+r^{-1}z''W^{\cdot\cdot} = 0. \quad (12)$$

Equations (7) and the homogeneous parts of equations (10), and equation (12) and the homogeneous part of equation (11) exhibit the static-geometric analogy of shell theory [1] in a form appropriate for shallow shells:

$$(M_r, M_\theta, M_{r\theta}, N_r, N_\theta, N_{r\theta}, F) \leftrightarrow (\varepsilon_\theta, \varepsilon_r, -\varepsilon_{r\theta}, -\kappa_\theta, -\kappa_r, \kappa_{r\theta}, W). \quad (13)$$

Upon substituting equations (7) and (9) into equation (11), and equations (8) and (10) into equation (12), and assuming $A = A(r)$, $D = D(r)$, and ν constant, the following two coupled equations for W and F are obtained

$$\nabla^2(D\nabla^2 W)-(1-\nu)K(D, W)-K(z, F) = P_z+r^{-1}(rz'\Phi)', \quad (14)$$

$$\nabla^2(A\nabla^2 F)-(1+\nu)K(A, F)+K(z, W) = -(1-\nu)\nabla^2(A\Phi). \quad (15)$$

In this ∇^2 is the two-dimensional Laplace operator in polar coordinates and

$$K(D, W) = \frac{D'}{r}W''+\frac{D''}{r}W'+\frac{D}{r^2}W^{\cdot\cdot}. \quad (16)$$

For A, D, and h constant, and $\Phi = 0$, equations (14) and (15) reduce to the Marguerre shallow shell equations [3].

At this point we could introduce the assumption of lateral deformation and attempt a direct integration of equations (14) and (15). However the significance of the two constants of integration which are introduced thereby is not immediately apparent.

Furthermore, it is known that F, in general, must include a multi-valued portion, F_m. (See [4], for example, for a discussion of the need of a multi-valued stress function for a spherical shell subject to a lateral side force.) It is not clear from equations (14) and (15) however, what the explicit form of F_m should be.

To avoid these two difficulties, we consider, in the next Section, integrals representing overall equilibrium and compatibility of the shell which can be determined without reference to equations (14) and (15). When lateral deformations are assumed, certain of these integrals become non-trivial first integrals of equations (14) and (15) with associated constants of integration having immediate physical interpretation, while others lead to the explicit form of F_m.

INTEGRAL CONDITIONS

The stress resultants and couples must, for arbitrary deformations, satisfy six integral relations which express the overall force and moment equilibrium of the shell. Six additional integral relations involving the extensional and bending strains, which may be interpreted as displacement and rotation continuity conditions, can be found from the static-geometric analogy.

Of interest for what follows are the equations of overall force equilibrium along the x-axis and overall moment equilibrium about the y-axis (see Fig. 1), together with the two analogous geometric conditions.

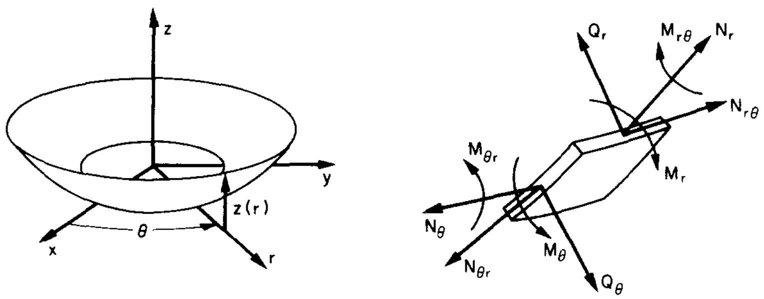

FIG. 1. Midsurface geometry and stress conventions.

The force and moment equations read

$$P(r)+\int_0^{2\pi}\int_a^r (P_r\cos\theta - P_\theta\sin\theta) r\, dr d\theta = P_0 \tag{17}$$

and

$$z(r)P(r)+M(r)+\int_0^{2\pi}\int_a^r [z(P_r\cos\theta-P_\theta\sin\theta)-rP_z\cos\theta] r\, dr d\theta = M_0 \tag{18}$$

where a is the radius of the inner edge of the shell,

$$P(r) = r\int_0^{2\pi}(N_r\cos\theta - N_{r\theta}\sin\theta)\,d\theta \tag{19}$$

is the net lateral force along the x-axis acting on the horizontal section z = const.

$$M(r) = r\int_0^{2\pi}[M_r\cos\theta - M_{r\theta}\sin\theta - r(Q_r+z'N_r)\cos\theta]\,d\theta \tag{20}$$

is the net moment acting on the horizontal section z = const. calculated about a line parallel to the y-axis and lying in the plane of the section; and where P_0 and M_0 denote the values of the x-component of the force and the y-component of the moment at the origin, $z = 0$.

By equation (13) the two geometric analogs of equations (17) and (18) are:

$$\int_0^{2\pi}(\kappa_\theta\cos\theta+\kappa_{r\theta}\sin\theta)\,d\theta = 0, \tag{21}$$

$$\int_0^{2\pi}\{\varepsilon_\theta\cos\theta+\varepsilon_{r\theta}\sin\theta+[-(r\varepsilon_\theta)'+\varepsilon_{r\theta}^{\cdot}+\varepsilon_r+rz'\kappa_\theta]\cos\theta\}\,d\theta = 0. \tag{22}$$

For future use, we note the form assumed by equations (17) and (21) when the variables involved are expressed in terms of W and F. Inserting equations (10) into equation (17) and expressing P_r and P_θ in terms of Φ, we get

$$P(r) = r^{-1}\Big[F^{\cdot}(r,2\pi)-F^{\cdot}(r,0)+r\int_0^{2\pi}\Phi\cos\theta\,d\theta\Big] \tag{23}$$

which leads to the relation

$$F^{\cdot}(r, 2\pi) - F^{\cdot}(r, 0) = r\left[P_0 - a\int_0^{2\pi} \Phi(a, \theta)\cos\theta \, d\theta\right]. \tag{24}$$

Equation (24) shows that, in general, part of the solution for F is multi-valued, and that this multi-valued part is proportional to r. The geometric analog of equation (24), which follows upon inserting equations (7) into equation (21), is simply

$$W^{\cdot}(r, 2\pi) - W^{\cdot}(r, 0) = 0$$

which, for continuous displacements and rotations, is identically satisfied.

Since we have assumed that $A = A(r)$, $D = D(r)$, $\nu = \text{const.}$, and that the shell is complete in the θ-direction, it follows that, for arbitrary deformations, P_z, Φ, the mid-surface displacements, the stress resultants and couples, and the *single-valued* part of F, F_s, can all be expanded in Fourier series of the form

$$\sum_{n=0}^{\infty} a_n(r)\cos n\theta + b_n(r)\sin n\theta.$$

When solutions of this form are substituted into the six equations of overall force and moment equilibrium, the two equations of axial force and axial moment equilibrium are found to depend on the $n = 0$ components only, while the remaining four equations of force and moment equilibrium along and about the x- and y-axes are found to depend on the $n = 1$ components only. The $n \geq 2$ components do not contribute to the overall force and moment integrals since they lead to a system of self-equilibrating loads; similar statements hold for the six analogous geometric integrals. Thus the study of the solutions to the governing equations reduces to a study of the three cases* $n = 0$, $n = 1$, and $n \geq 2$. The remainder of this note is devoted to the applications of these integral relations to the $n = 1$ case.

SIMPLIFICATION OF THE DIFFERENTIAL EQUATIONS FOR $n = 1$

For $n = 1$, we consider solutions of the form

$$(F_s, W, U, N_r, N_\theta, Q_r, M_r, M_\theta, \Phi, P_z) = (f_s, w, \ldots, p_z)\cos\theta, \tag{25}$$

$$(V, N_{r\theta}, Q_\theta, M_{r\theta}) = (v, n_{r\theta}, q_\theta, m_{r\theta})\sin\theta, \tag{26}$$

which are associated with the two integrals of force equilibrium along the x-axis and moment equilibrium about the y-axis given by equations (17) and (18). It is sufficient to consider solutions of this form since the solutions associated with the integrals of force equilibrium along and about the y- and x-axes, respectively, can be obtained from the above solutions by replacing θ by $\theta + \pi/2$.

We first determine the explicit form of F_m, the multi-valued part of F. We have already noted, from equation (24), that F_m must be proportional to r. But, for $n = 1$, $(N_r, N_\theta) = (n_r, n_\theta)\cos\theta$ and $N_{r\theta} = n_{r\theta}\sin\theta$. Hence, it follows from equation (10) that

$$F_m = Br\theta\sin\theta \tag{27}$$

* For a shell of revolution with two edges fixed to rigid circular inserts to which forces and moments are applied, we note that the complete solution is composed of the $n = 0$ and $n = 1$ solutions alone.

and substituting equation (27) into equation (24) we get

$$B = \frac{P_0}{2\pi} - \frac{a}{2}\phi(a). \tag{28}$$

We now consider the differential equations satisfied by $w(r)$ and $f_s(r)$. The basic simplification for $n = 1$ is that, instead of equations (14) and (15), equations (18) and (22) may be taken as differential equations for w and f_s. *Indeed, for $n = 1$, equations* (18) *and* (22) *are nothing more than first integrals of equations* (14) *and* (15), *respectively*. A second simplification follows from the fact that $w = r$ and $f_s = r$ must be solutions of the homogeneous form of these differential equations since they yield zero stress resultants and bending strains. Altogether then, we obtain upon setting

$$\begin{aligned} W &= w(r)\cos\theta \\ &= r\left(\int \chi\,\mathrm{d}r + C_1\right)\cos\theta \end{aligned} \tag{29}$$

and

$$\begin{aligned} F &= F_m(r,\ \theta) + f_s(r)\cos\theta \\ &= r\left\{\tfrac{1}{2}[P_0/\pi - a\phi(a)]\theta\sin\theta + \left(\int\psi\,\mathrm{d}r + C_2\right)\cos\theta\right\} \end{aligned} \tag{30}$$

and substituting these expressions into equations (18) and (22), the following two *second-order* differential equations for χ and ψ:

$$\begin{aligned} r^2\chi'' + \left(\frac{rD'}{D}+3\right)r\chi' + \left[(2+\nu)\frac{rD'}{D} - 3\right]\chi - \frac{z'r}{D}\psi \\ = \frac{M_0}{\pi D r} + \frac{rP_0}{\pi D}\left(\frac{z}{r}\right)' + \frac{a\phi(a)}{Dr}[z(r)-z(a)] + \frac{z'}{D}[r\phi(r) - a\phi(a)] \\ + \frac{1}{Dr}\int_a^r (rp_z - z'\phi)r\,\mathrm{d}r, \end{aligned} \tag{31}$$

$$\begin{aligned} r^2\psi'' + \left(\frac{rA'}{A}+3\right)r\psi' + \left[(2-\nu)\frac{rA'}{A} - 3\right]\psi + \frac{z'r}{A}\chi \\ = \left[\frac{P_0}{\pi} - a\phi(a)\right]\left[\frac{1-\nu}{r} - \nu\frac{A'}{A}\right] - (1-\nu)\left[r\frac{(A\phi)'}{A} - \phi\right]. \end{aligned} \tag{32}$$

Equations (31) and (32) are of comparable simplicity to the two coupled second-order equations one obtains for axi-symmetric deformations [5].

EXPLICIT SOLUTION OF THE DIFFERENTIAL EQUATIONS FOR A CLASS OF SHELLS

We consider now shells with constant thickness and elastic properties whose meridional slope varies as a power of the radius:

$$z'(r) = \alpha\rho^s, \qquad \rho = r/a. \tag{33}$$

Furthermore, for simplicity, we restrict ourselves to edge loaded shells ($\phi = p_z = 0$), take ρ as the new independent variable, and now use primes to denote differentiation with respect to ρ. Equations (31) and (32) then reduce to

$$\begin{aligned}\rho^2\chi''+3\rho\chi'-3\chi-\left(\frac{\alpha a}{D}\right)\rho^{s+1}\psi \\ = \frac{M_0}{\pi D a\rho}+\frac{\alpha P_0}{\pi D}\left(\rho^s-\frac{1}{\rho}\int\rho^s\,\mathrm{d}\rho\right)\end{aligned} \tag{34}$$

$$\rho^2\psi''+3\rho\psi'-3\psi+\left(\frac{\alpha a}{A}\right)\rho^{s+1}\chi = \frac{(1-\nu)P_0}{\pi a\rho}. \tag{35}$$

Equations (34) and (35) may be further reduced to a single complex equation. Multiplying equation (35) by $i(A/D)^{\frac{1}{2}}$ and adding it to equation (34), we obtain

$$\begin{aligned}\left[\rho^2\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}+3\rho\frac{\mathrm{d}}{\mathrm{d}\rho}-3+i\frac{\alpha a}{\sqrt{(AD)}}\rho^{s+1}\right]\left[\chi+i\sqrt{\left(\frac{A}{D}\right)}\psi\right] \\ = \frac{1}{\pi a\rho}\left[\frac{M_0}{D}+i(1-\nu)\sqrt{\left(\frac{A}{D}\right)}P_0\right]+\frac{\alpha P_0}{\pi D}\left(\rho^s-\frac{1}{\rho}\int\rho^s\mathrm{d}\rho\right).\end{aligned} \tag{36}$$

To discuss the solutions of equation (36), we must distinguish the two cases $s \neq -1$, and $s = -1$, the latter corresponding to a shell of logarithmic profile.

For $s \neq -1$, the operator on the left-hand side of equation (36) is a Bessel operator, and the complete solutions for χ and ψ can be written

$$\chi = \frac{1}{\rho}[C_3 ber_m(m\mu\rho^{2/m})+C_4 ker_m(m\mu\rho^{2/m})]-\frac{s}{s+1}\frac{P_0}{\pi a\rho}+\chi_p, \tag{37}$$

$$\psi = \frac{1}{\rho}\sqrt{\left(\frac{A}{D}\right)}[C_3 bei_m(m\mu\rho^{2/m})+C_4 kei_m(m\mu\rho^{2/m})]+\psi_\rho, \tag{38}$$

where

$$m = \frac{4}{s+1}, \qquad 4\mu^2 = \frac{\alpha a}{\sqrt{(AD)}},$$

and where χ_ρ and ψ_ρ are particular solutions expressible in terms of Lommel functions [6, p. 350].* For shallow spherical ($s = 1$) and conical ($s = 0$) shells, equations (37) and (38) reduce to known results [4, 7].

For $s = -1$, the operator on the left-hand side of equation (36) is equidimensional, and the complete solution is

$$\chi+i\sqrt{\left(\frac{A}{D}\right)}\psi = C_3\rho^{p_1}+C_4\rho^{p_2}-\frac{1+i\mu^2}{4\pi D(1+\mu^4)\rho}\left\{\frac{M_0}{a}+\alpha P_0\left[1-\ln\rho+i\left(\frac{1-\nu}{4\mu^4}\right)\right]\right\} \tag{39}$$

where C_3 and C_4 are arbitrary complex constants and p_1 and p_2 are the two roots of

$$p^2+2p-3-i4\mu^2 = 0.$$

* We note that for axi-symmetric deflections ($n = 0$), the homogeneous solutions of the corresponding second-order differential equation have exactly the same form as the homogeneous parts of equations (37) and (38) but with $m = 2/(s+1)$, [5].

This solution in powers of ρ (for $P_0 = M_0 = 0$) is a special case of the solution for all values of the integer n obtained in [8].

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Zusammenfassung—Es werden Drehschalen mit radial veränderlicher Dicke untersucht, welche dem Sinus und Kosinus des Polarwinkels proportionalen Deformationen (seitlichen Deformationen) unterliegen. Durch die Verwendung gewisser Intergrale, welche das Gesamtgleichgewicht und die Kompatibilität der Schale zum Ausdruck bringen, lässt es sich zeigen, dass die Lösung der bestimmenden Differentialgleichung auf die Lösung zweier gekoppelter Differentialgleichungen zweiter Ordnung reduziert werden kann. Die expliziten Lösungen für eine Schalenklasse, deren meridionale Neigung sich mit der Potenz des Radius ändert, werden in Kelvinfunktionen und deren Integralen ausgedrückt.

Абстракт—Рассматриваются неглубокие оболочки тел вращения с радиально изменяющейся толщиной, подверженные небольшой деформации, пропорциональной синусу или косинусу полярного угла (боковая деформация).

Применением известных интегралов, выражающих общее равновесие и совместность оболочки, демонстрируется что решение управляющих дифференциальных уравнений может быть сведено к решению двух спаренных дифференциальных уравнений второго порядка. Явные решения для того класса оболочек, у которого меридиональный уклон изменяется со степенью радиуса, получаются в условиях функций Кельвина и их интегралов.